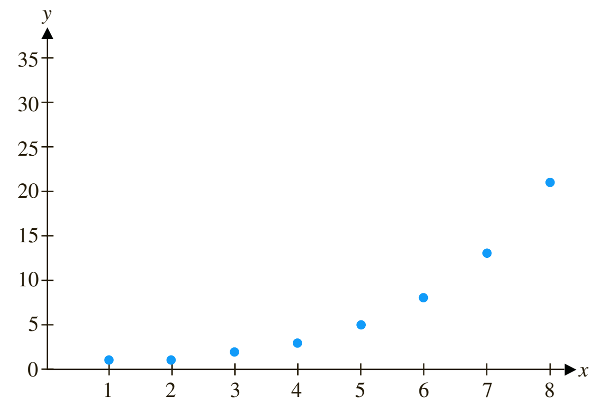


FIGURE 0.1
The Fibonacci sequence

these is the attempt to find patterns to help us better describe the world. The other theme is the interplay between graphs and functions. By connecting the powerful equation-solving techniques of algebra with the visual images provided by graphs, you will significantly improve your ability to make use of your mathematical skills in solving real-world problems.

0.1 POLYNOMIALS AND RATIONAL FUNCTIONS

The Real Number System and Inequalities

Although mathematics is far more than just a study of numbers, our journey into calculus begins with the real number system. While this may seem to be a fairly mundane starting place, we want to give you the opportunity to brush up on those properties that are of particular interest for calculus.

The most familiar set of numbers is the set of **integers,** consisting of the whole numbers and their additive inverses: $0, \pm 1, \pm 2, \pm 3, \ldots$. A **rational number** is any number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, $\frac{2}{3}$, $-\frac{7}{3}$ and $\frac{27}{125}$ are all rational numbers. Notice that every integer n is also a rational number, since we can write it as the quotient of two integers: $n = \frac{n}{1}$.

The **irrational numbers** are all those real numbers that cannot be written in the form $\frac{p}{q}$, where p and q are integers. Recall that rational numbers have decimal expansions that either terminate or repeat. For instance, $\frac{1}{2} = 0.5$, $\frac{1}{3} = 0.3333\bar{3}$, $\frac{1}{8} = 0.125$ and $\frac{1}{6} = 0.16666\bar{6}$ are all rational numbers. By contrast, irrational numbers have decimal expansions that do not repeat or terminate. For instance, three familiar irrational numbers and their decimal expansions are

$$\sqrt{2} = 1.41421\,35623\ldots,$$
$$\pi = 3.14159\,26535\ldots$$

and

$$e = 2.71828\,18284\ldots.$$

We picture the real numbers arranged along the number line displayed in Figure 0.2 (the **real line**). The set of real numbers is denoted by the symbol $\mathbb{R}$.

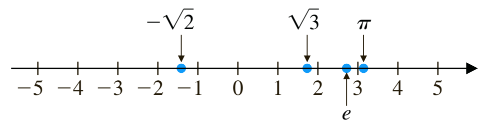


FIGURE 0.2
The real line

For real numbers a and b, where $a < b$, we define the **closed interval** $[a, b]$ to be the set of numbers between a and b, including a and b (the **endpoints**), that is,

$$[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\},$$

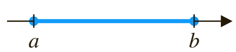


FIGURE 0.3
A closed interval

as illustrated in Figure 0.3, where the solid circles indicate that a and b are included in $[a, b]$.

Similarly, the **open interval** (a, b) is the set of numbers between a and b, but *not* including the endpoints a and b, that is,

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\},$$

FIGURE 0.4
An open interval

as illustrated in Figure 0.4, where the open circles indicate that a and b are not included in (a, b).

You should already be very familiar with the following properties of real numbers.

THEOREM 1.1

If a and b are real numbers and $a < b$, then

(i) For any real number c, $a + c < b + c$.
(ii) For real numbers c and d, if $c < d$, then $a + c < b + d$.
(iii) For any real number $c > 0$, $a \cdot c < b \cdot c$.
(iv) For any real number $c < 0$, $a \cdot c > b \cdot c$.

REMARK 1.1

We need the properties given in Theorem 1.1 to solve inequalities. Notice that (i) says that you can add the same quantity to both sides of an inequality. Part (iii) says that you can multiply both sides of an inequality by a positive number. Finally, (iv) says that if you multiply both sides of an inequality by a negative number, the inequality is reversed.

We illustrate the use of Theorem 1.1 by solving a simple inequality.

EXAMPLE 1.1 Solving a Linear Inequality

Solve the linear inequality $2x + 5 < 13$.

Solution We can use the properties in Theorem 1.1 to isolate the x. First, subtract 5 from both sides to obtain

$$(2x + 5) - 5 < 13 - 5$$

or
$$2x < 8.$$

Finally, divide both sides by 2 (since $2 > 0$, the inequality is not reversed) to obtain

$$x < 4.$$

We often write the solution of an inequality in interval notation. In this case, we get the interval $(-\infty, 4)$. ■

You can deal with more complicated inequalities in the same way.

EXAMPLE 1.2 Solving a Two-Sided Inequality

Solve the two-sided inequality $6 < 1 - 3x \leq 10$.

Solution First, recognize that this problem requires that we find values of x such that

$$6 < 1 - 3x \quad \textit{and} \quad 1 - 3x \leq 10.$$

Here, we can use the properties in Theorem 1.1 to isolate the x by working on both inequalities simultaneously. First, subtract 1 from each term, to get

$$6 - 1 < (1 - 3x) - 1 \leq 10 - 1$$

or
$$5 < -3x \leq 9.$$

Now, divide by -3, but be careful. Since $-3 < 0$, the inequalities are reversed. We have

$$\frac{5}{-3} > \frac{-3x}{-3} \geq \frac{9}{-3}$$

or
$$-\frac{5}{3} > x \geq -3.$$

We usually write this as
$$-3 \leq x < -\frac{5}{3},$$

or in interval notation as $[-3, -\frac{5}{3})$. ■

You will often need to solve inequalities involving fractions. We present a typical example in the following.

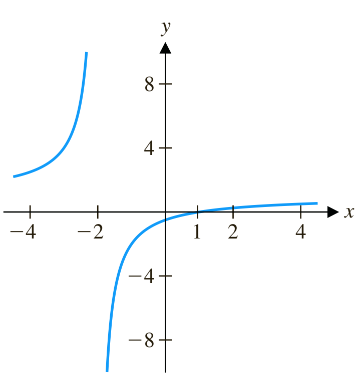
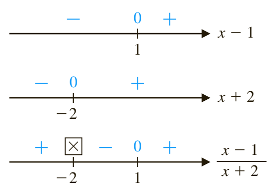


FIGURE 0.5
$$y = \frac{x - 1}{x + 2}$$

EXAMPLE 1.3 Solving an Inequality Involving a Fraction

Solve the inequality $\dfrac{x - 1}{x + 2} \geq 0$.

Solution In Figure 0.5, we show a graph of the function, which appears to indicate that the solution includes all $x < -2$ and $x \geq 1$. Carefully read the inequality and observe that there are only three ways to satisfy this: either both numerator and denominator are positive, both are negative or the numerator is zero. To visualize this, we draw number lines for each of the individual terms, indicating where each is positive, negative or zero and use these to draw a third number line indicating the value of the quotient, as shown in the margin. In the third number line, we have placed an "⊠"



above the -2 to indicate that the quotient is undefined at $x = -2$. From this last number line, you can see that the quotient is nonnegative whenever $x < -2$ *or* $x \geq 1$. We write the solution in interval notation as $(-\infty, -2) \cup [1, \infty)$. ■

For inequalities involving a polynomial of degree 2 or higher, factoring the polynomial and determining where the individual factors are positive and negative, as in example 1.4, will lead to a solution.

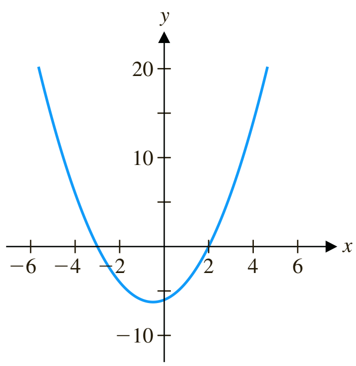
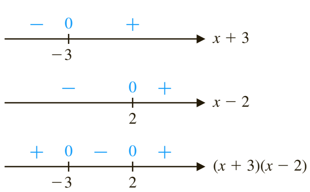


FIGURE 0.6
$y = x^2 + x - 6$



EXAMPLE 1.4 Solving a Quadratic Inequality

Solve the quadratic inequality

$$x^2 + x - 6 > 0. \tag{1.1}$$

Solution In Figure 0.6, we show a graph of the polynomial on the left side of the inequality. Since this polynomial factors, (1.1) is equivalent to

$$(x + 3)(x - 2) > 0. \tag{1.2}$$

This can happen in only two ways: when both factors are positive or when both factors are negative. As in example 1.3, we draw number lines for both of the individual factors, indicating where each is positive, negative or zero and use these to draw a number line representing the product. We show these in the margin. Notice that the third number line indicates that the product is positive whenever $x < -3$ *or* $x > 2$. We write this in interval notation as $(-\infty, -3) \cup (2, \infty)$. ■

No doubt, you will recall the following standard definition.

DEFINITION 1.1

The **absolute value** of a real number x is $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$.

Make certain that you read Definition 1.1 correctly. If x is negative, then $-x$ is positive. This says that $|x| \geq 0$ for all real numbers x. For instance, using the definition,

$$|-4| = -(-4) = 4,$$

notice that for any real numbers a and b,

$$|a \cdot b| = |a| \cdot |b|.$$

However,
$$|a + b| \neq |a| + |b|,$$

in general. (To verify this, simply take $a = 5$ and $b = -2$ and compute both quantities.)

However, it is always true that

$$|a + b| \leq |a| + |b|.$$

This is referred to as the **triangle inequality.**

The interpretation of $|a - b|$ as the distance between a and b (see the note in the margin) is particularly useful for solving inequalities involving absolute values. Wherever possible, we suggest that you use this interpretation to read what the inequality means, rather than merely following a procedure to produce a solution.

NOTES

For any two real numbers a and b, $|a - b|$ gives the *distance* between a and b. (See Figure 0.7.)

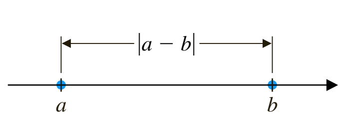


FIGURE 0.7
The distance between a and b

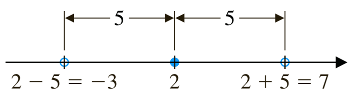


FIGURE 0.8
$|x-2|<5$

EXAMPLE 1.5 Solving an Inequality Containing an Absolute Value

Solve the inequality

$$|x-2|<5. \tag{1.3}$$

Solution Before you start trying to solve this, take a few moments to read what it *says*. Since $|x-2|$ gives the distance from x to 2, (1.3) says that the *distance* from x to 2 must be *less than* 5. So, find all numbers x whose distance from 2 is less than 5. We indicate the set of all numbers within a distance 5 of 2 in Figure 0.8. You can now read the solution directly from the figure: $-3<x<7$ or in interval notation: $(-3, 7)$. ■

Many inequalities involving absolute values can be solved simply by reading the inequality correctly, as in example 1.6.

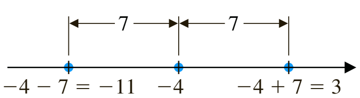


FIGURE 0.9
$|x+4|\le 7$

EXAMPLE 1.6 Solving an Inequality with a Sum Inside an Absolute Value

Solve the inequality

$$|x+4|\le 7. \tag{1.4}$$

Solution To use our distance interpretation, we must first rewrite (1.4) as

$$|x-(-4)|\le 7.$$

This now says that the distance from x to -4 is less than or equal to 7. We illustrate the solution in Figure 0.9, from which it follows that $-11\le x\le 3$ or $[-11, 3]$. ■

Recall that for any real number $r>0$, $|x|<r$ is equivalent to the following inequality not involving absolute values:

$$-r<x<r.$$

In example 1.7, we use this to revisit the inequality from example 1.5.

EXAMPLE 1.7 An Alternative Method for Solving Inequalities

Solve the inequality $|x-2|<5$.

Solution This is equivalent to the two-sided inequality

$$-5<x-2<5.$$

Adding 2 to each term, we get the solution

$$-3<x<7,$$

or in interval notation $(-3, 7)$, as before. ■

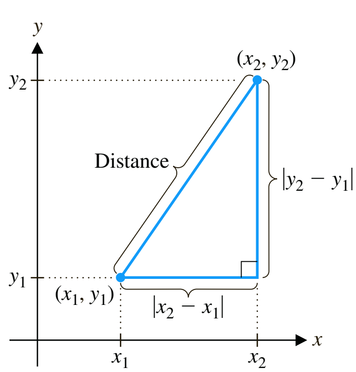


FIGURE 0.10
Distance

Recall that the distance between two points (x_1, y_1) and (x_2, y_2) is a simple consequence of the Pythagorean Theorem and is given by

$$d\{(x_1, y_1), (x_2, y_2)\}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

We illustrate this in Figure 0.10.

EXAMPLE 1.8 Using the Distance Formula

Find the distance between the points (1, 2) and (3, 4).

Solution The distance between (1, 2) and (3, 4) is

$$d\{(1,2),(3,4)\} = \sqrt{(3-1)^2+(4-2)^2} = \sqrt{4+4} = \sqrt{8}. \blacksquare$$

Equations of Lines

Year	*U.S. Population*
1960	179,323,175
1970	203,302,031
1980	226,542,203
1990	248,709,873

x	y
0	179
10	203
20	227
30	249

Transformed data

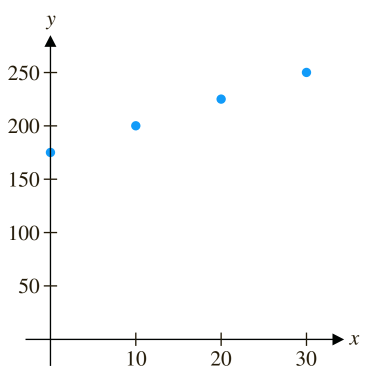


FIGURE 0.11
Population data

The federal government conducts a nationwide census every 10 years to determine the population. Population data for the last several decades are shown in the accompanying table.

One difficulty with analyzing these data is that the numbers are so large. This problem is remedied by **transforming** the data. We can simplify the year data by defining x to be the number of years since 1960. Then, 1960 corresponds to $x = 0$, 1970 corresponds to $x = 10$ and so on. The population data can be simplified by rounding the numbers to the nearest million. The transformed data are shown in the accompanying table and a scatter plot of these data points is shown in Figure 0.11.

Most people would say that the points in Figure 0.11 appear to form a straight line. (Use a ruler and see if you agree.) To determine whether the points are, in fact, on the same line (such points are called **colinear**), we might consider the population growth in each of the indicated decades. From 1960 to 1970, the growth was 24 million. (That is, to move from the first point to the second, you increase x by 10 and increase y by 24.) Likewise, from 1970 to 1980, the growth was 24 million. However, from 1980 to 1990, the growth was only 22 million. Since the rate of growth is not constant, the data points do not fall on a line. Notice that to stay on the same line, y would have had to increase by 24 again. The preceding argument involves the familiar concept of *slope*.

DEFINITION 1.2

For $x_1 \neq x_2$, the **slope** of the straight line through the points (x_1, y_1) and (x_2, y_2) is the number

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1.5}$$

When $x_1 = x_2$, the line through (x_1, y_1) and (x_2, y_2) is **vertical** and the slope is undefined.

We often describe slope as "the change in y divided by the change in x," written $\frac{\Delta y}{\Delta x}$, or more simply as $\frac{\text{Rise}}{\text{Run}}$ (see Figure 0.12a on the following page).

The slope of a straight line is the same no matter which two points on the line you select. Referring to Figure 0.12b (where the line has positive slope), notice that for any four points A, B, D and E on the line, the two right triangles ΔABC and ΔDEF are similar. Recall that for similar triangles, the ratios of corresponding sides must be the same. In this case, this says that

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y'}{\Delta x'}$$

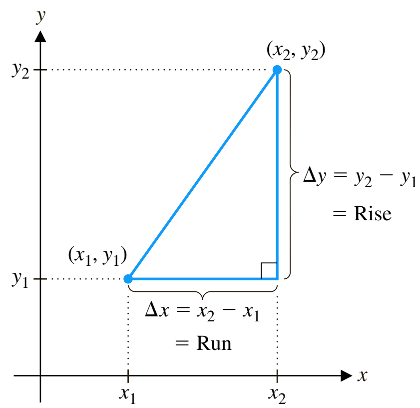


FIGURE 0.12a
Slope

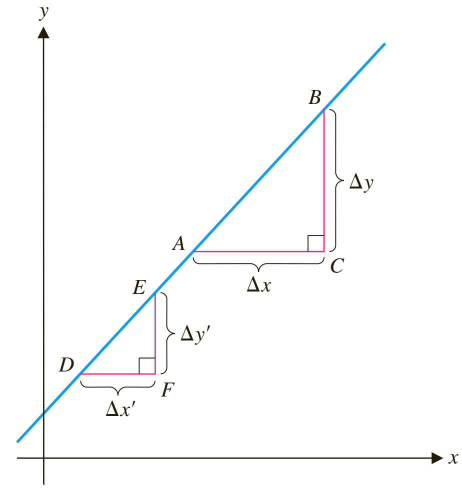


FIGURE 0.12b
Similar triangles and slope

and so, the slope is the same no matter which two points on the line are selected. Furthermore, a line is the only curve with constant slope. Notice that a line is **horizontal** if and only if its slope is zero.

EXAMPLE 1.9 Finding the Slope of a Line

Find the slope of the line through the points (4, 3) and (2, 5).

Solution From (1.5), we get

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{2-4} = \frac{2}{-2} = -1. \blacksquare$$

EXAMPLE 1.10 Using Slope to Determine if Points Are Colinear

Use slope to determine whether the points (1, 2), (3, 10) and (4, 14) are colinear.

Solution First, notice that the slope of the line joining (1, 2) and (3, 10) is

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10-2}{3-1} = \frac{8}{2} = 4.$$

Similarly, the slope through the line joining (3, 10) and (4, 14) is

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14-10}{4-3} = 4.$$

Since the slopes are the same, the points must be colinear. ■

Recall that if you know the slope and a point through which the line must pass, you have enough information to graph the line. The easiest way to graph a line is to plot two points and then draw the line through them. In this case, you need only to find a second point.

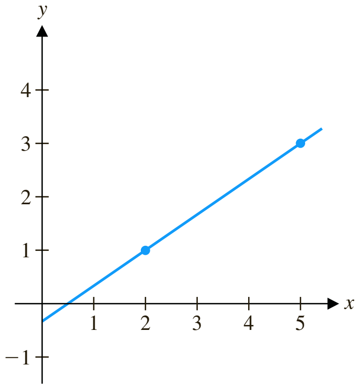


FIGURE 0.13a
Graph of straight line

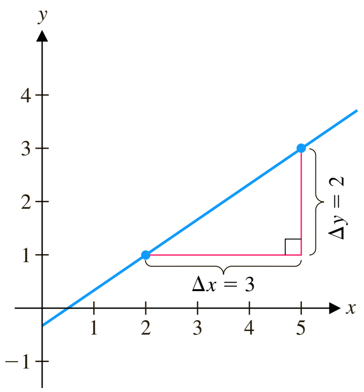


FIGURE 0.13b
Using slope to find a second point

EXAMPLE 1.11 Graphing a Line

If a line passes through the point (2, 1) with slope $\frac{2}{3}$, find a second point on the line and then graph the line.

Solution Since slope is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, we take $m = \frac{2}{3}$, $y_1 = 1$ and $x_1 = 2$, to obtain

$$\frac{2}{3} = \frac{y_2 - 1}{x_2 - 2}.$$

You are free to choose the x-coordinate of the second point. For instance, to find the point at $x_2 = 5$, substitute this in and solve. From

$$\frac{2}{3} = \frac{y_2 - 1}{5 - 2} = \frac{y_2 - 1}{3},$$

we get $2 = y_2 - 1$ or $y_2 = 3$. A second point is then (5, 3). The graph of the line is shown in Figure 0.13a. An alternative method for finding a second point is to use the slope

$$m = \frac{2}{3} = \frac{\Delta y}{\Delta x}.$$

The slope of $\frac{2}{3}$ says that if we move three units to the right, we must move two units up to stay on the line, as illustrated in Figure 0.13b. ■

In example 1.11, the choice of $x = 5$ was entirely arbitrary; you can choose any x-value you want to find a second point. Further, since x can be any real number, you can leave x as a variable and write out an equation satisfied by any point (x, y) on the line. In the general case of the line through the point (x_0, y_0) with slope m, we have from (1.5) that

$$m = \frac{y - y_0}{x - x_0}. \tag{1.6}$$

Multiplying both sides of (1.6) by $(x - x_0)$, we get

$$y - y_0 = m(x - x_0)$$

or

POINT-SLOPE FORM OF A LINE

$$y = m(x - x_0) + y_0. \tag{1.7}$$

Equation (1.7) is called the **point-slope form** of the line.

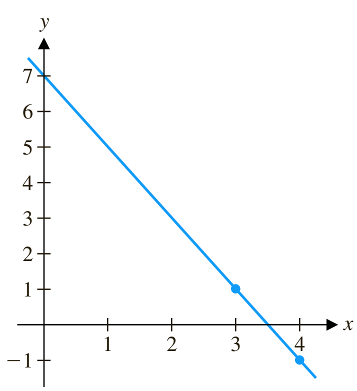


FIGURE 0.14
$y = -2(x - 3) + 1$

EXAMPLE 1.12 Finding the Equation of a Line Given Two Points

Find an equation of the line through the points (3, 1) and (4, −1), and graph the line.

Solution From (1.5), the slope is $m = \dfrac{-1 - 1}{4 - 3} = \dfrac{-2}{1} = -2$. Using (1.7) with slope $m = -2$, x-coordinate $x_0 = 3$ and y-coordinate $y_0 = 1$, we get the equation of the line:

$$y = -2(x - 3) + 1. \tag{1.8}$$

To graph the line, plot the points (3, 1) and (4, −1), and you can easily draw the line seen in Figure 0.14. ■

In example 1.12, you may be tempted to simplify the expression for y given in (1.8). As it turns out, the point-slope form of the equation is often the most convenient to work with. So, we will typically not ask you to rewrite this expression in other forms. At times, a form of the equation called the **slope-intercept form** is more convenient. This has the form

$$y = mx + b,$$

where m is the slope and b is the y-intercept (i.e., the place where the graph crosses the y-axis). In example 1.12, you simply multiply out (1.8) to get $y = -2x + 6 + 1$ or

$$y = -2x + 7.$$

As you can see from Figure 0.14, the graph crosses the y-axis at $y = 7$.

Theorem 1.2 presents a familiar result on parallel and perpendicular lines.

THEOREM 1.2

Two (nonvertical) lines are **parallel** if they have the same slope. Further, any two vertical lines are parallel. Two (nonvertical) lines of slope m_1 and m_2 are **perpendicular** whenever the product of their slopes is -1 (i.e., $m_1 \cdot m_2 = -1$). Also, any vertical line and any horizontal line are perpendicular.

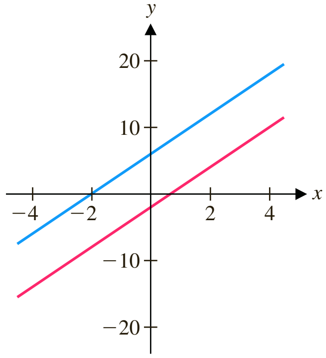


FIGURE 0.15
Parallel lines

Since we can read the slope from the equation of a line, it's a simple matter to determine when two lines are parallel or perpendicular. We illustrate this in examples 1.13 and 1.14.

EXAMPLE 1.13 Finding the Equation of a Parallel Line

Find an equation of the line parallel to $y = 3x - 2$ and through the point $(-1, 3)$.

Solution It's easy to read the slope of the line from the equation: $m = 3$. The equation of the parallel line is then

$$y = 3[x - (-1)] + 3$$

or simply $y = 3(x + 1) + 3$. We show a graph of both lines in Figure 0.15. ■

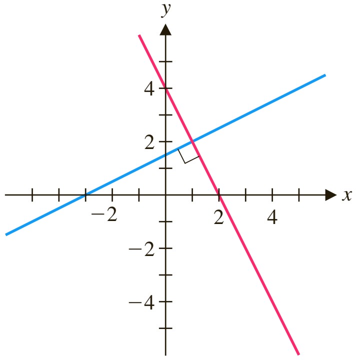


FIGURE 0.16
Perpendicular lines

EXAMPLE 1.14 Finding the Equation of a Perpendicular Line

Find an equation of the line perpendicular to $y = -2x + 4$ and intersecting the line at the point $(1, 2)$.

Solution The slope of $y = -2x + 4$ is -2. The slope of the perpendicular line is then $-1/(-2) = \frac{1}{2}$. Since the line must pass through the point $(1, 2)$, the equation of the perpendicular line is

$$y = \frac{1}{2}(x - 1) + 2.$$

We show a graph of the two lines in Figure 0.16. ■

We now return to this subsection's introductory example and use the equation of a line to estimate the population in the year 2000.

EXAMPLE 1.15 Using a Line to Estimate Population

Given the population data for the census years 1960, 1970, 1980 and 1990, estimate the population for the year 2000.

Solution We began this subsection by showing that the points in the corresponding table are not colinear. Nonetheless, they are *nearly* colinear. So, why not use the straight line connecting the last two points (20, 227) and (30, 249) (corresponding to the populations in the years 1980 and 1990) to estimate the population in 2000? (This is a simple example of a more general procedure called **extrapolation.**) The slope of the line joining the two data points is

$$m = \frac{249 - 227}{30 - 20} = \frac{22}{10} = \frac{11}{5}.$$

The equation of the line is then

$$y = \frac{11}{5}(x - 30) + 249.$$

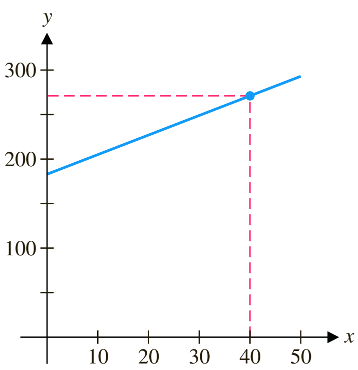


FIGURE 0.17
Population

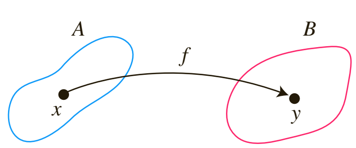
See Figure 0.17 for a graph of the line. If we follow this line to the point corresponding to $x = 40$ (the year 2000), we have the estimated population

$$\frac{11}{5}(40 - 30) + 249 = 271.$$

That is, the estimated population is 271 million people. The actual census figure for 2000 was 281 million, which indicates that the U.S. population has grown at a rate that is faster than linear. ■

Functions

For any two subsets A and B of the real line, we make the following familiar definition.



DEFINITION 1.3

A **function** f is a rule that assigns *exactly one* element y in a set B to each element x in a set A. In this case, we write $y = f(x)$.

We call the set A the **domain** of f. The set of all values $f(x)$ in B is called the **range** of f. That is, the range of f is $\{f(x) \mid x \in A\}$. Unless explicitly stated otherwise, the domain of a function f is the largest set of real numbers for which the function is defined. We refer to x as the **independent variable** and to y as the **dependent variable.**

REMARK 1.2

Functions can be defined by simple formulas, such as $f(x) = 3x + 2$, but in general, any correspondence meeting the requirement of matching exactly *one* y to each x defines a function.

By the **graph** of a function f, we mean the graph of the equation $y = f(x)$. That is, the graph consists of all points (x, y), where x is in the domain of f and where $y = f(x)$.

Notice that not every curve is the graph of a function, since for a function, only one y-value corresponds to a given value of x. You can graphically determine whether a curve is the graph of a function by using the **vertical line test:** if any vertical line intersects the graph in more than one point, the curve is not the graph of a function.

EXAMPLE 1.16 Using the Vertical Line Test

Determine which of the curves in Figures 0.18a and 0.18b correspond to functions.

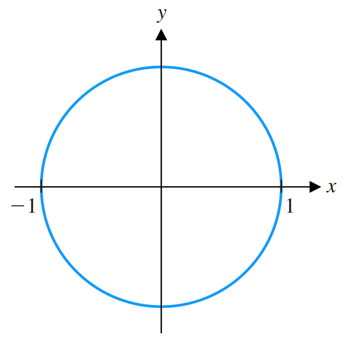


FIGURE 0.18a

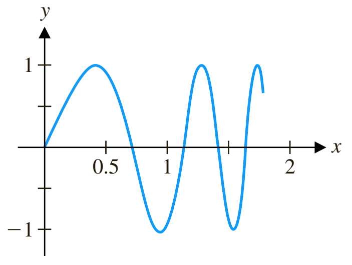


FIGURE 0.18b

Solution Notice that the circle in Figure 0.18a is not the graph of a function, since a vertical line at $x = 0.5$ intersects the circle twice (see Figure 0.19a). The graph in Figure 0.18b is the graph of a function, even though it swings up and down repeatedly. Although horizontal lines intersect the graph repeatedly, vertical lines, such as the one at $x = 1.2$, intersect only once (see Figure 0.19b). ■

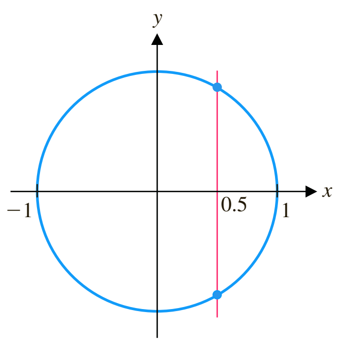


FIGURE 0.19a
Curve fails vertical line test

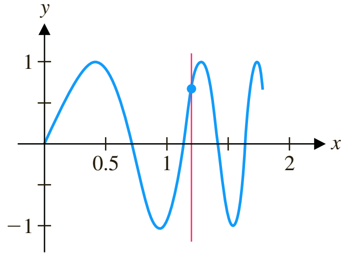


FIGURE 0.19b
Curve passes vertical line test

You are already familiar with a number of different types of functions, and we will only briefly review these here and in sections 0.4 and 0.5. The functions that you are probably most familiar with are *polynomials*. These are the simplest functions to work with because they are defined entirely in terms of arithmetic.

DEFINITION 1.4

A **polynomial** is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, a_2, \ldots, a_n$ are real numbers (the **coefficients** of the polynomial) with $a_n \neq 0$ and $n \geq 0$ is an integer (the **degree** of the polynomial).

Note that the domain of every polynomial function is the entire real line. Further, recognize that the graph of the linear (degree 1) polynomial $f(x) = ax + b$ is a straight line.

EXAMPLE 1.17 Sample Polynomials

The following are all examples of polynomials:

$f(x) = 2$ (polynomial of degree 0 or **constant**),

$f(x) = 3x + 2$ (polynomial of degree 1 or **linear** polynomial),

$f(x) = 5x^2 - 2x + 1$ (polynomial of degree 2 or **quadratic** polynomial),

$f(x) = x^3 - 2x + 1$ (polynomial of degree 3 or **cubic** polynomial),

$f(x) = -6x^4 + 12x^2 - 3x + 13$ (polynomial of degree 4 or **quartic** polynomial),

and

$$f(x) = 2x^5 + 6x^4 - 8x^2 + x - 3 \text{ (polynomial of degree 5 or } \textbf{quintic} \text{ polynomial).}$$

We show graphs of these six functions in Figures 0.20a–0.20f.

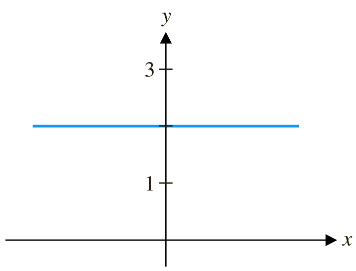


FIGURE 0.20a
$f(x) = 2$

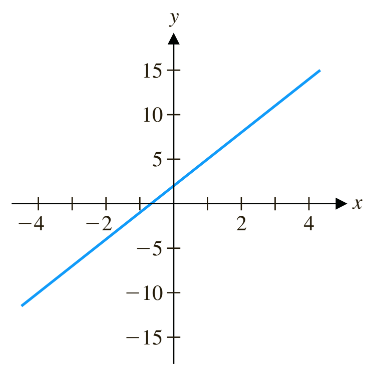


FIGURE 0.20b
$f(x) = 3x + 2$

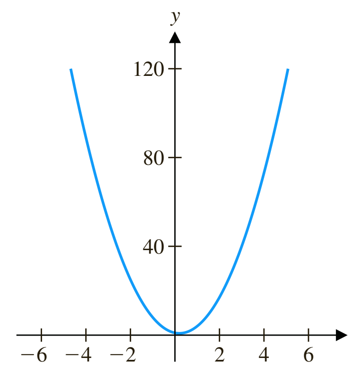


FIGURE 0.20c
$f(x) = 5x^2 - 2x + 1$

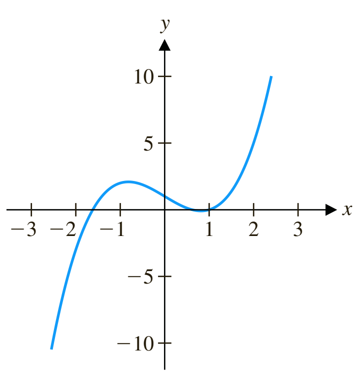


FIGURE 0.20d
$f(x) = x^3 - 2x + 1$

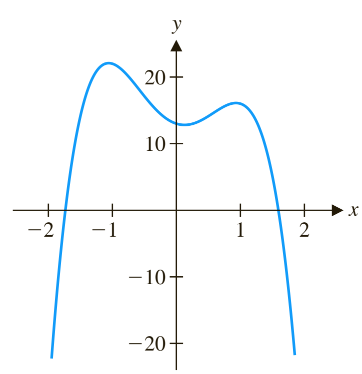


FIGURE 0.20e
$f(x) = -6x^4 + 12x^2 - 3x + 13$

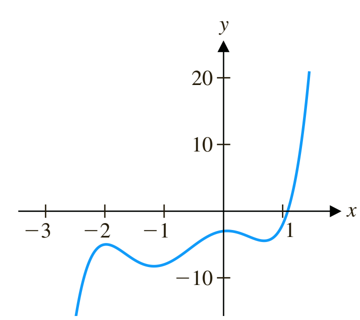


FIGURE 0.20f
$f(x) = 2x^5 + 6x^4 - 8x^2 + x - 3$

DEFINITION 1.5

Any function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomials, is called a **rational** function.

Notice that since $p(x)$ and $q(x)$ are polynomials, they are both defined for all x, and so, the rational function $f(x) = \dfrac{p(x)}{q(x)}$ is defined for all x for which $q(x) \neq 0$.

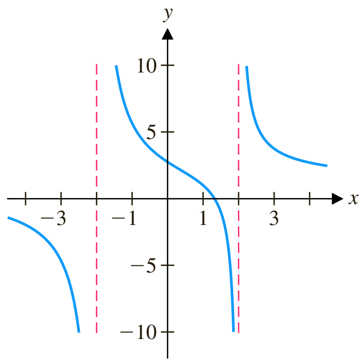


FIGURE 0.21
$f(x) = \dfrac{x^2 + 7x - 11}{x^2 - 4}$

EXAMPLE 1.18 A Sample Rational Function

Find the domain of the function

$$f(x) = \frac{x^2 + 7x - 11}{x^2 - 4}.$$

Solution Here, $f(x)$ is a rational function. We show a graph in Figure 0.21. Its domain consists of those values of x for which the denominator is nonzero. Notice that

$$x^2 - 4 = (x - 2)(x + 2)$$

and so, the denominator is zero if and only if $x = \pm 2$. This says that the domain of f is

$$\{x \in \mathbb{R} \,|\, x \neq \pm 2\} = (-\infty, -2) \cup (-2, 2) \cup (2, \infty). \blacksquare$$

The **square root** function is defined in the usual way. When we write $y = \sqrt{x}$, we mean that y is that number for which $y^2 = x$ and $y \geq 0$. In particular, $\sqrt{4} = 2$. Be careful not to write erroneous statements such as $\sqrt{4} = \pm 2$. In particular, be careful to write

$$\sqrt{x^2} = |x|.$$

Since $\sqrt{x^2}$ is asking for the *nonnegative* number whose square is x^2, we are looking for $|x|$ and not x. We can say

$$\sqrt{x^2} = x, \text{ } \textit{only} \text{ for } x \geq 0.$$

Similarly, for any integer $n \geq 2$, $y = \sqrt[n]{x}$ whenever $y^n = x$, where for n even, $x \geq 0$ and $y \geq 0$.

EXAMPLE 1.19 Finding the Domain of a Function Involving a Square Root or a Cube Root

Find the domains of $f(x) = \sqrt{x^2 - 4}$ and $g(x) = \sqrt[3]{x^2 - 4}$.

Solution Since even roots are defined only for nonnegative values, $f(x)$ is defined only for $x^2 - 4 \geq 0$. Notice that this is equivalent to having $x^2 \geq 4$, which occurs when $x \geq 2$ or $x \leq -2$. The domain of f is then $(-\infty, -2] \cup [2, \infty)$. On the other hand, odd roots are defined for both positive and negative values. Consequently, the domain of $g(x)$ is the entire real line, $(-\infty, \infty)$. $\blacksquare$

We often find it useful to label intercepts and other significant points on a graph. Finding these points typically involves solving equations. A solution of the equation $f(x) = 0$ is called a **zero** of the function f or a **root** of the equation $f(x) = 0$. Notice that a zero of the function f corresponds to an x-intercept of the graph of $y = f(x)$.

EXAMPLE 1.20 Finding Zeros by Factoring

Find all x- and y-intercepts of $f(x) = x^2 - 4x + 3$.

Solution To find the y-intercept, set $x = 0$ to obtain

$$y = 0 - 0 + 3 = 3.$$

To find the x-intercepts, solve the equation $f(x) = 0$. In this case, we can factor to get

$$f(x) = x^2 - 4x + 3 = (x - 1)(x - 3) = 0.$$

You can now read off the zeros: $x = 1$ and $x = 3$, as indicated in Figure 0.22. $\blacksquare$

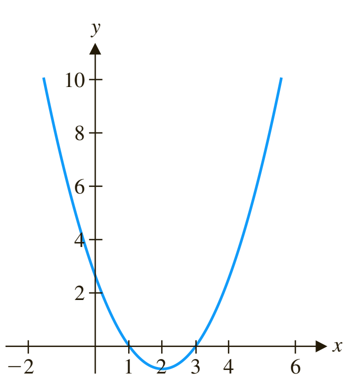


FIGURE 0.22
$y = x^2 - 4x + 3$

Unfortunately, factoring is not always so easy. Of course, for the quadratic equation

$$ax^2 + bx + c = 0$$

(for $a \neq 0$), the solution(s) are given by the familiar **quadratic formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 1.21 Finding Zeros Using the Quadratic Formula

Find the zeros of $f(x) = x^2 - 5x - 12$.

Solution You probably won't have much luck trying to factor this. However, from the quadratic formula, we have

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot (-12)}}{2 \cdot 1} = \frac{5 \pm \sqrt{25 + 48}}{2} = \frac{5 \pm \sqrt{73}}{2}.$$

So, the two solutions are given by $x = \frac{5}{2} + \frac{\sqrt{73}}{2} \approx 6.772$ and $x = \frac{5}{2} - \frac{\sqrt{73}}{2} \approx -1.772$. (No wonder you couldn't factor the polynomial!) ■

Finding zeros of polynomials of degree higher than 2 and other functions is usually trickier and is sometimes impossible. At the least, you can always find an approximation of any zero(s) by using a graph to zoom in closer to the point(s) where the graph crosses the x-axis, as we'll illustrate shortly. A more basic question, though, is to determine *how many* zeros a given function has. In general, there is no way to answer this question without the use of calculus. For the case of polynomials, however, Theorem 1.3 (a consequence of the Fundamental Theorem of Algebra) provides a clue.

THEOREM 1.3

A polynomial of degree n has *at most* n distinct zeros.

REMARK 1.3

Polynomials may also have complex zeros. For instance, $f(x) = x^2 + 1$ has only the complex zeros $x = \pm i$, where i is the imaginary number defined by $i = \sqrt{-1}$.

Notice that Theorem 1.3 does not say how many zeros a given polynomial has, but rather, that the *maximum* number of distinct (i.e., different) zeros is the same as the degree. A polynomial of degree n may have anywhere from 0 to n distinct real zeros. However, polynomials of odd degree must have *at least one* real zero. For instance, for the case of a cubic polynomial, we have one of the three possibilities illustrated in Figures 0.23a, 0.23b and 0.23c on the following page.

In these three figures, we show the graphs of cubic polynomials with 1, 2 and 3 distinct, real zeros, respectively. These are the graphs of the functions

$$f(x) = x^3 - 2x^2 + 3 = (x + 1)(x^2 - 3x + 3),$$
$$g(x) = x^3 - x^2 - x + 1 = (x + 1)(x - 1)^2$$

and

$$h(x) = x^3 - 3x^2 - x + 3 = (x + 1)(x - 1)(x - 3),$$

respectively. Note that you can see from the factored form where the zeros are (and how many there are).

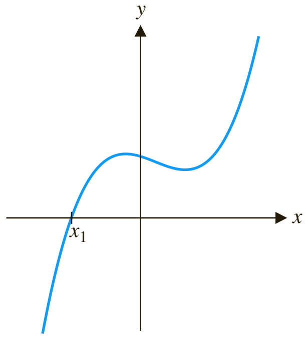


FIGURE 0.23a
One zero

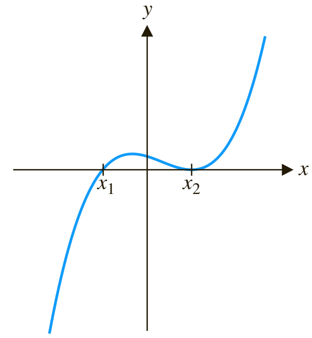


FIGURE 0.23b
Two zeros

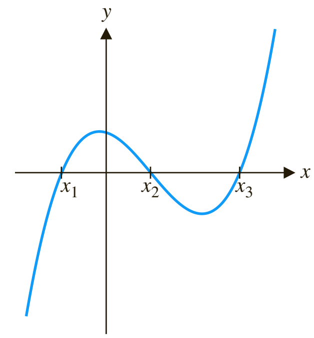


FIGURE 0.23c
Three zeros

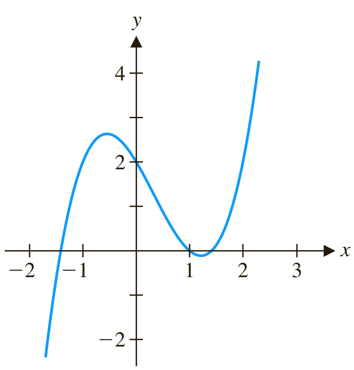


FIGURE 0.24a
$y = x^3 - x^2 - 2x + 2$

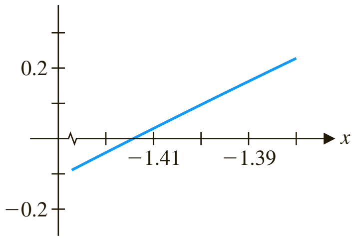


FIGURE 0.24b
Zoomed in on zero near $x = -1.4$

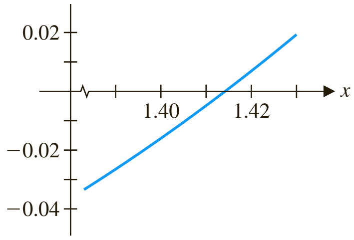


FIGURE 0.24c
Zoomed in on zero near $x = 1.4$

Theorem 1.4 provides an important connection between factors and zeros of polynomials.

THEOREM 1.4 (Factor Theorem)
For any polynomial f, $f(a) = 0$ if and only if $(x - a)$ is a factor of $f(x)$.

EXAMPLE 1.22 Finding the Zeros of a Cubic Polynomial

Find the zeros of $f(x) = x^3 - x^2 - 2x + 2$.

Solution By calculating $f(1)$, you can see that one zero of this function is $x = 1$, but how many other zeros are there? A graph of the function (see Figure 0.24a) shows that there are two other zeros of f, one near $x = -1.5$ and one near $x = 1.5$. You can find these zeros more precisely by using your graphing calculator or computer algebra system to zoom in on the locations of these zeros (as shown in Figures 0.24b and 0.24c). From these zoomed graphs it is clear that the two remaining zeros of f are near $x = 1.41$ and $x = -1.41$. You can make these estimates more precise by zooming in even more closely. Most graphing calculators and computer algebra systems can also find approximate zeros, using a built-in "solve" program. In Chapter 3, we present a versatile method (called Newton's method) for obtaining accurate approximations to zeros. The only way to find the exact solutions is to factor the expression (using either long division or synthetic division). Here, we have

$$f(x) = x^3 - x^2 - 2x + 2 = (x-1)(x^2-2) = (x-1)(x-\sqrt{2})(x+\sqrt{2}),$$

from which you can see that the zeros are $x = 1$, $x = \sqrt{2}$ and $x = -\sqrt{2}$. ■

Recall that to find the points of intersection of two curves defined by $y = f(x)$ and $y = g(x)$, we set $f(x) = g(x)$ to find the x-coordinates of any points of intersection.

EXAMPLE 1.23 Finding the Intersections of a Line and a Parabola

Find the points of intersection of the parabola $y = x^2 - x - 5$ and the line $y = x + 3$.

Solution A sketch of the two curves (see Figure 0.25) shows that there are two intersections, one near $x = -2$ and the other near $x = 4$. To determine these precisely,

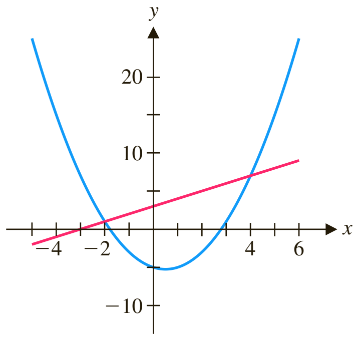


FIGURE 0.25
$y = x + 3$ and $y = x^2 - x - 5$

we set the two functions equal and solve for x:

$$x^2 - x - 5 = x + 3.$$

Subtracting $(x + 3)$ from both sides leaves us with

$$0 = x^2 - 2x - 8 = (x - 4)(x + 2).$$

This says that the solutions are exactly $x = -2$ and $x = 4$. We compute the corresponding y-values from the equation of the line $y = x + 3$ (or the equation of the parabola). The points of intersection are then $(-2, 1)$ and $(4, 7)$. Notice that these are consistent with the intersections seen in Figure 0.25. ■

Unfortunately, you won't always be able to solve equations exactly, as we did in examples 1.20–1.23. We explore some options for dealing with more difficult equations in section 0.2.

EXERCISES 0.1

WRITING EXERCISES

1. If the slope of the line passing through points A and B equals the slope of the line passing through points B and C, explain why the points A, B and C are colinear.
2. If a graph fails the vertical line test, it is not the graph of a function. Explain this result in terms of the definition of a function.
3. You should not automatically write the equation of a line in slope-intercept form. Compare the following forms of the same line: $y = 2.4(x - 1.8) + 0.4$ and $y = 2.4x - 3.92$. Given $x = 1.8$, which equation would you rather use to compute y? How about if you are given $x = 0$? For $x = 8$, is there any advantage to one equation over the other? Can you quickly read off the slope from either equation? Explain why neither form of the equation is "better."
4. To understand Definition 1.1, you must believe that $|x| = -x$ for negative x's. Using $x = -3$ as an example, explain in words why multiplying x by -1 produces the same result as taking the absolute value of x.

In exercises 1–4, determine if the points are colinear.

1. $(2, 1), (0, 2), (4, 0)$
2. $(3, 1), (4, 4), (5, 8)$
3. $(4, 1), (3, 2), (1, 3)$
4. $(1, 2), (2, 5), (4, 8)$

In exercises 5–10, find the slope of the line through the given points.

5. $(1, 2), (3, 6)$
6. $(1, 2), (3, 3)$
7. $(3, -6), (1, -1)$
8. $(1, -2), (-1, -3)$
9. $(0.3, -1.4), (-1.1, -0.4)$
10. $(1.2, 2.1), (3.1, 2.4)$

In exercises 11–16, find a second point on the line with slope *m* and point *P*, graph the line and find an equation of the line.

11. $m = 2$, $P = (1, 3)$
12. $m = -2$, $P = (1, 4)$
13. $m = 0$, $P = (-1, 1)$
14. $m = \frac{1}{2}$, $P = (2, 1)$
15. $m = 1.2$, $P = (2.3, 1.1)$
16. $m = -\frac{1}{4}$, $P = (-2, 1)$

In exercises 17–22, determine if the lines are parallel, perpendicular, or neither.

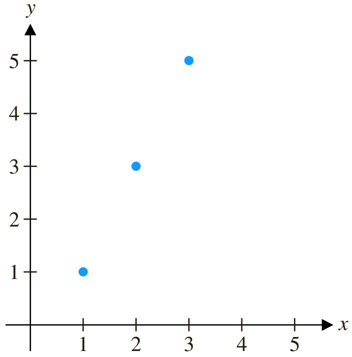
17. $y = 3(x - 1) + 2$ and $y = 3(x + 4) - 1$
18. $y = 2(x - 3) + 1$ and $y = 4(x - 3) + 1$
19. $y = -2(x + 1) - 1$ and $y = \frac{1}{2}(x - 2) + 3$
20. $y = 2x - 1$ and $y = -2x + 2$
21. $y = 3x + 1$ and $y = -\frac{1}{3}x + 2$
22. $x + 2y = 1$ and $2x + 4y = 3$

In exercises 23–26, find an equation of a line through the given point and (a) parallel to and (b) perpendicular to the given line.

23. $y = 2(x + 1) - 2$ at $(2, 1)$
24. $y = 3(x - 2) + 1$ at $(0, 3)$
25. $y = 2x + 1$ at $(3, 1)$
26. $y = 1$ at $(0, -1)$

In exercises 27–30, find an equation of the line through the given points and compute the y-coordinate of the point on the line corresponding to $x = 4$.

27.



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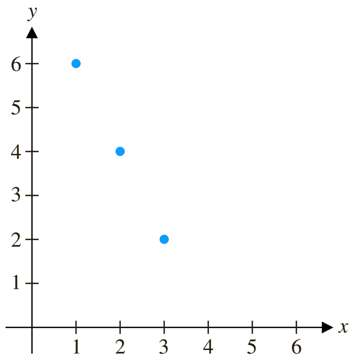

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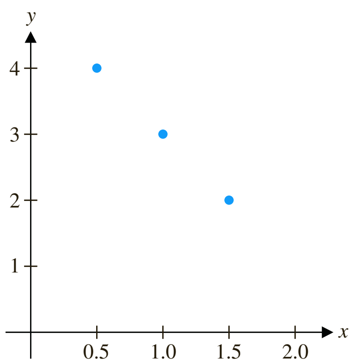

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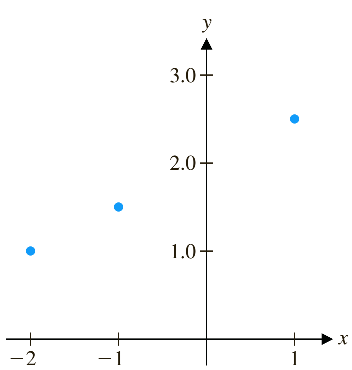
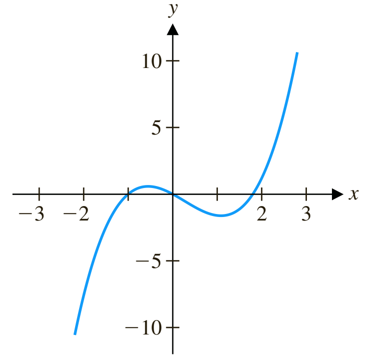


In exercises 31–34, use the vertical line test to determine whether the curve is the graph of a function.

31.



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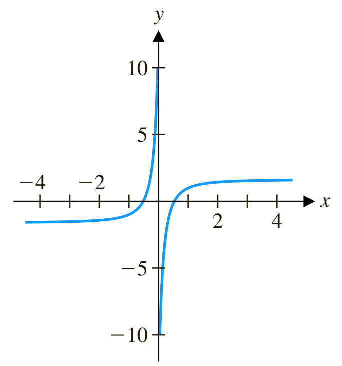

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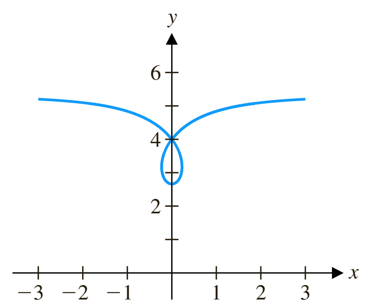

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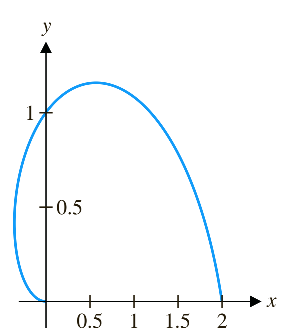

In exercises 35–40, identify the given function as polynomial, rational, both or neither.

35. $f(x) = x^3 - 4x + 1$

36. $f(x) = 3 - 2x + x^4$

37. $f(x) = \dfrac{x^2 + 2x - 1}{x + 1}$

38. $f(x) = \dfrac{x^3 + 4x - 1}{x^4 - 1}$

39. $f(x) = \sqrt{x^2 + 1}$

40. $f(x) = 2x - x^{2/3} - 6$

In exercises 41–46, find the domain of the function.

41. $f(x) = \sqrt{x + 2}$

42. $f(x) = \sqrt{2x + 1}$

43. $f(x) = \sqrt[3]{x - 1}$

44. $f(x) = \sqrt{x^2 - 4}$

45. $f(x) = \dfrac{4}{x^2 - 1}$

46. $f(x) = \dfrac{4x}{x^2 + 2x - 6}$

In exercises 47–50, find the indicated function values.

47. $f(x) = x^2 - x - 1;\quad f(0), f(2), f(-3), f(1/2)$

48. $f(x) = \dfrac{x + 1}{x - 1};\quad f(0), f(2), f(-2), f(1/2)$

49. $f(x) = \sqrt{x + 1};\quad f(0), f(3), f(-1), f(1/2)$

50. $f(x) = \dfrac{3}{x};\quad f(1), f(10), f(100), f(1/3)$

In exercises 51–54, a brief description is given of a physical situation. For the indicated variable, state a reasonable domain.

51. A parking deck is to be built; x = width of deck (in feet).

52. A parking deck is to be built on a 200′-by-200′ lot; x = width of deck (in feet).

53. A new candy bar is to be sold; x = number of candy bars sold in the first month.

54. A new candy bar is to be sold; x = cost of candy bar (in cents).

In exercises 55–58, discuss whether you think y would be a function of x.

55. y = grade you get on an exam, x = number of hours you study

56. y = probability of getting lung cancer, x = number of cigarettes smoked per day

57. y = a person's weight, x = number of minutes exercising per day

58. y = speed at which an object falls, x = weight of object

59. Figure A shows the speed of a bicyclist as a function of time. For the portions of this graph that are flat, what is happening to the bicyclist's speed? What is happening to the bicyclist's speed when the graph goes up? down? Identify the portions of the graph that correspond to the bicyclist going uphill; downhill.

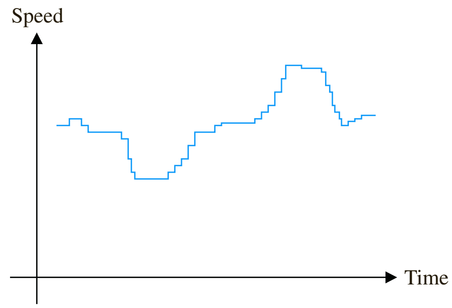

FIGURE A
Bicycle speed

60. Figure B shows the population of a small country as a function of time. During the time period shown, the country experienced two influxes of immigrants, a war and a plague. Identify these important events.

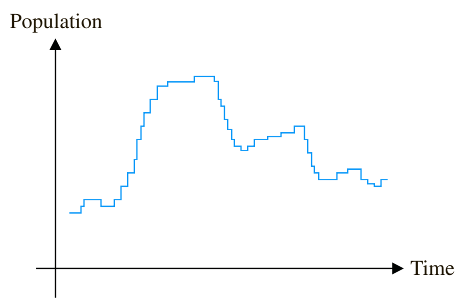

FIGURE B
Population

In exercises 61–66, find all intercepts of the given graph.

61. $y = x^2 - 2x - 8$

62. $y = x^2 + 4x + 4$

63. $y = x^3 - 8$

64. $y = x^3 - 3x^2 + 3x - 1$

65. $y = \dfrac{x^2 - 4}{x + 1}$

66. $y = \dfrac{2x - 1}{x^2 - 4}$

In exercises 67–74, factor and/or use the quadratic formula to find all zeros of the given function.

67. $f(x) = x^2 - 4x + 3$

68. $f(x) = x^2 + x - 12$

69. $f(x) = x^2 - 4x + 2$

70. $f(x) = 2x^2 + 4x - 1$

71. $f(x) = x^3 - 3x^2 + 2x$

72. $f(x) = x^3 - 2x^2 - x + 2$

73. $f(x) = x^6 + x^3 - 2$

74. $f(x) = x^3 + x^2 - 4x - 4$

75. The boiling point of water (in degrees Fahrenheit) at elevation h (in thousands of feet above sea level) is given by $B(h) = -1.8h + 212$. Find h such that water boils at 98.6°. Why would this altitude be dangerous to humans?

76. The spin rate of a golf ball hit with a 9 iron has been measured at 9100 rpm for a 120-compression ball and at 10,000 rpm for a 60-compression ball. Most golfers use 90-compression balls. If the spin rate is a linear function of compression, find the spin rate for a 90-compression ball. Professional golfers often use 100-compression balls. Estimate the spin rate of a 100-compression ball.

77. The chirping rate of a cricket depends on the temperature. A species of tree cricket chirps 160 times per minute at 79°F and 100 times per minute at 64°F. Find a linear function relating temperature to chirping rate.

78. When describing how to measure temperature by counting cricket chirps, most guides suggest that you count the number of chirps in a 15-second time period. Use exercise 77 to explain why this is a convenient period of time.

79. A person has played a computer game many times. The statistics show that she has won 415 times and lost 120 times, and the winning percentage is listed as 78%. How many times in a row must she win to raise the reported winning percentage to 80%?

EXPLORATORY EXERCISES

1. Suppose you have a machine that will proportionally enlarge a photograph. For example, it could enlarge a 4×6 photograph to 8×12 by doubling the width and height. You could make an 8×10 picture by cropping 1 inch off each side. Explain how you would enlarge a $3\frac{1}{2} \times 5$ picture to an 8×10. A friend returns from Scotland with a $3\frac{1}{2} \times 5$ picture showing the Loch Ness monster in the outer $\frac{1}{4}''$ on the right. If you use your procedure to make an 8×10 enlargement, does Nessie make the cut?

2. Solve the equation $|x-2| + |x-3| = 1$. (Hint: It's an unusual solution, in that it's more than just a couple of numbers.) Then, solve the equation $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$. (Hint: If you make the correct substitution, you can use your solution to the previous equation.)

0.2 GRAPHING CALCULATORS AND COMPUTER ALGEBRA SYSTEMS

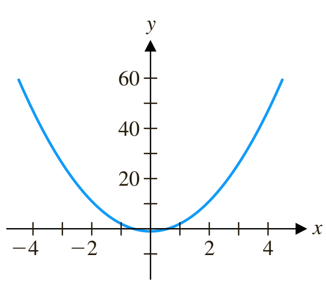

FIGURE 0.26a
$y = 3x^2 - 1$

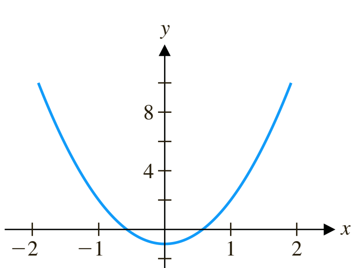

FIGURE 0.26b
$y = 3x^2 - 1$

The relationships between functions and their graphs are central topics in calculus. Graphing calculators and user-friendly computer software allow you to explore these relationships for a much wider variety of functions than you could with pencil and paper alone. This section presents a general framework for using technology to explore the graphs of functions.

Recall that the graphs of linear functions are straight lines and the graphs of quadratic polynomials are parabolas. One of the goals of this section is for you to become more familiar with the graphs of other functions. The best way to become familiar is through experience, by working example after example.

EXAMPLE 2.1 Generating a Calculator Graph

Use your calculator or computer to sketch a graph of $f(x) = 3x^2 - 1$.

Solution You should get an initial graph that looks something like that in Figure 0.26a. This is simply a parabola opening upward. A graph is often used to search for important points, such as x-intercepts, y-intercepts or peaks and troughs. In this case, we could see these points better if we zoom in, that is, display a smaller range of x- and y-values than the technology has initially chosen for us. The graph in Figure 0.26b shows x-values from $x = -2$ to $x = 2$ and y-values from $y = -2$ to $y = 10$.

You can see more clearly in Figure 0.26b that the parabola bottoms out roughly at the point $(0, -1)$ and crosses the x-axis at approximately $x = -0.5$ and $x = 0.5$. You can make this more precise by doing some algebra. Recall that an x-intercept is a point where $y = 0$ or $f(x) = 0$. Solving $3x^2 - 1 = 0$ gives $3x^2 = 1$ or $x^2 = \frac{1}{3}$, so that $x = \pm\sqrt{\frac{1}{3}} \approx \pm 0.57735$. ■